Exercise 7.2.18

Solve the ODE

$$(x^2 - y^2 e^{y/x})\,dx + (x^2 + xy)e^{y/x}\,dy = 0.$$

Hint. Note that the quantity y/x in the exponents is of combined degree zero and does not affect the determination of homogeneity.

Solution

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(x^2 - y^2 e^{y/x}) \neq \frac{\partial}{\partial x}[(x^2 + xy)e^{y/x}]$$
$$-\frac{ye^{y/x}}{x}(2x + y) \neq \frac{e^{y/x}}{x}(2x^2 - y^2).$$

In order to make it so, multiply both sides of the ODE by an integrating factor I.

$$(x^2 - y^2 e^{y/x})I\,dx + (x^2 + xy)e^{y/x}I\,dy = 0$$

Now that it is exact, we have

$$\frac{\partial}{\partial y}[(x^2 - y^2 e^{y/x})I] = \frac{\partial}{\partial x}[(x^2 + xy)e^{y/x}I]$$
$$-\frac{ye^{y/x}}{x}(2x + y)I + (x^2 - y^2 e^{y/x})\frac{\partial I}{\partial y} = \frac{e^{y/x}}{x}(2x^2 - y^2)I + (x^2 + xy)e^{y/x}\frac{\partial I}{\partial x}.$$

To solve for a simple integrating factor, assume that it's only a function of x: $I = I(x)$.

$$-\frac{ye^{y/x}}{x}(2x + y)I = \frac{e^{y/x}}{x}(2x^2 - y^2)I + (x^2 + xy)e^{y/x}\frac{dI}{dx}$$

Divide both sides by $e^{y/x}$.

$$-\frac{1}{x}(2xy + y^2)I = \frac{1}{x}(2x^2 - y^2)I + (x^2 + xy)\frac{dI}{dx}$$
$$0 = \frac{1}{x}(2x^2 + 2xy)I + (x^2 + xy)\frac{dI}{dx}$$
$$0 = 2(x + y)I + x(x + y)\frac{dI}{dx}$$

Divide both sides by $x + y$.

$$0 = 2I + x\frac{dI}{dx}$$
$$0 = 2 + x\frac{\frac{dI}{dx}}{I}$$
$$\frac{\frac{dI}{dx}}{I} = -\frac{2}{x}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dx}\ln|I| = -\frac{2}{x}$$

Integrate both sides with respect to x.

$$\begin{aligned}\ln|I| &= -2\ln|x| + C_1\\ &= \ln x^{-2} + C_1\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}|I| &= e^{\ln x^{-2}+C_1}\\ &= e^{\ln x^{-2}}e^{C_1}\\ &= x^{-2}e^{C_1}\end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right.

$$I(x) = \pm e^{C_1}x^{-2}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$I(x) = C_2x^{-2}$$

Any integrating factor will do, so choose $C_2 = 1$ for the simplest.

$$I(y) = x^{-2}$$

Now that the integrating factor is known, the original ODE can be solved.

$$(x^2 - y^2e^{y/x})\,dx + (x^2 + xy)e^{y/x}\,dy = 0$$

Multiply both sides by x^{-2}.

$$\left(1 - \frac{y^2}{x^2}e^{y/x}\right)dx + \left(1 + \frac{y}{x}\right)e^{y/x}\,dy = 0 \tag{1}$$

Since it's exact now, there exists a potential function $\varphi = \varphi(x, y)$ that satisfies

$$\frac{\partial\varphi}{\partial x} = 1 - \frac{y^2}{x^2}e^{y/x} \tag{2}$$

$$\frac{\partial\varphi}{\partial y} = \left(1 + \frac{y}{x}\right)e^{y/x}. \tag{3}$$

As a result, equation (1) can be written as

$$\frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy = 0.$$

The left side is how the differential of φ is defined.

$$d\varphi = 0$$

Integrate both sides.

$$\varphi(x, y) = C_3$$

The general solution to the ODE is found then by solving equations (2) and (3) for φ. Integrate both sides of equation (3) partially with respect to y to get φ.

$$\varphi(x, y) = ye^{y/x} + f(x)$$

Differentiate both sides with respect to x.

$$\frac{\partial \varphi}{\partial x} = -\frac{y^2}{x^2} e^{y/x} + f'(x)$$

Comparing this formula for $\partial\varphi/\partial x$ with equation (2), we see that

$$f'(x) = 1.$$

Integrate both sides with respect to x.

$$f(x) = x + C_4$$

Therefore, the potential function is

$$\varphi(x, y) = ye^{y/x} + x + C_4,$$

and the general solution to the ODE is

$$ye^{y/x} + x = A,$$

where A is a new constant used for $C_3 - C_4$.